Connexivity in coherence-based probability logic

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A basic connexive intuition (overviews on connexive logic: McCall, 2012; Wansing, 2020)

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Under the material conditional interpretation of conditionals, however, it holds that:

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We cover the basic connexive intuition by the observation that for any event A, with $\bar{A} \neq \emptyset$, event $A|\bar{A}$ is

$$P(A|\bar{A}) = 0,$$

where 0 is the only coherent value.

Framework: Coherence-based probability logic

- Coherence (subjective probability)
  - de Finetti, and {Capotorti, Coletti, Gilio, Holzer, Lad, Regazzini, Rigo, Sanfilippo, Scozzafava, Vantaggi, Walley, ...}
  - probability as a degree of belief
  - in betting terms, a probability assessment is coherent if and only if in any finite combination of n bets it cannot happen that the values of the random gain are all positive, or all negative (no Dutch Book)
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  - zero probabilities are exploited to reduce the complexity
  - imprecision
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  - logical operations on conditional events (without triviality)
- Probability logic
  - uncertain argument forms
  - deductive consequence relation
  - **propagation of the uncertainties from the premises to the conclusions**

Motivation

"If two people are arguing 'If A will C?' and are both in doubt as to A, they are adding A hypothetically to their stock of knowledge and arguing on that basis about C; so that in a sense 'If A, C' and 'If A, $\bar{C}$' are contradictories. We can say they are fixing their degrees of belief in C given A. If A turns out false, these degrees of belief are rendered void."

(Ramsey, 1929/1994, p. 55, we adjusted the notation)

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"[A]ncient logicians most likely meant their theses [concerning connexivity] as applicable only to 'normal' conditionals with antecedents which are not self-contradictory." ***(Lenzen, 2020, p. 16)***

Our motivation

- Interpreting and checking the validity of connexive principles within coherence based probability logic
- Two approaches:
  - Approach 1: Connexive principles obtained by defaults interpreted by probabilistic constraints on conditional events (basic conditionals are interpreted by probability constraints; negation of the assessment; wide scope negation)
  - Approach 2: Connexive principles obtained by logical operations on conditional events (basic conditionals are interpreted by conditional events; negation of the object; narrow scope negation)

Conditionals as conditional events

A conditional *if H then A* that satisfies $P(\textit{if } H \textit{ then } A) = P(A|H)$ is identified with the conditional event $A|H$, with $H \neq \emptyset$, defined as a three-valued logical entity

$$A|H = \begin{cases} \text{True}, & \text{if} \quad A \wedge H = AH \quad \text{is true;} \\ \text{False}, & \text{if} \quad \bar{A}H \quad \text{is true;} \\ \text{Void}, & \text{if} \quad \bar{H} \quad \text{is true.} \end{cases}$$

We observe that $(A \wedge H)|H = A|H$.

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In terms of the betting metaphor, assessing $p = P(A|H)$ implies that you agree

$$\begin{array}{lcl} \text{to pay} & & \text{in order to receive} \\ \text{(resp., to receive)} & p & \text{(resp., to pay)} \end{array} \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \bar{A}H \text{ is true,} \\ p, & \text{if } \bar{H} \text{ is true.} \end{cases}$$

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Coherence requires that $P(A \wedge H) = P(A|H)P(H)$. When $P(H) > 0$ it follows that $P(A|H) = \frac{P(A \wedge H)}{P(H)}$. If $P(H) = 0$, it follows that $P(A \wedge H) = 0$; in this case $P(A|H) \in [0, 1]$. Moreover $P(A|A) = 1$ and $P(\bar{A}|A) = 0$.

Connexive principles

We investigate the following connexive principles (see also Wansing (2020) for the terminology):

- Aristotle's Thesis (AT): $\sim(\sim A \rightarrow A)$,
- Aristotle's Thesis' (AT'): $\sim(A \rightarrow \sim A)$,
- Abelard's First Principle (AB): $\sim((A \rightarrow B) \wedge (A \rightarrow \sim B))$,
- Aristotle's Second Thesis (AS): $\sim((A \rightarrow B) \wedge (\sim A \rightarrow B))$,
- Boethius' Thesis (BT): $(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$,
- Boethius' Thesis' (BT'): $(A \rightarrow \sim B) \rightarrow \sim(A \rightarrow B)$,
- Reversed Boethius' Thesis (RBT): $\sim(A \rightarrow \sim B) \rightarrow (A \rightarrow B)$,
- Reversed Boethius' Thesis' (RBT'): $\sim(A \rightarrow B) \rightarrow (A \rightarrow \sim B)$,
- Boethius Variation 3 (B3): $(A \rightarrow B) \rightarrow \sim(\sim A \rightarrow B)$,
- Boethius Variation 4 (B4): $(\sim A \rightarrow B) \rightarrow \sim(A \rightarrow B)$.

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Approach 1: Connexive principles and default reasoning

Here, we interpret the basic conditional $A \to C$ by the default $A \mid\!\sim C$, which is interpreted by the probability assessment $P(C|A) = 1$.

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Here, we interpret the basic conditional $A \to C$ by the default $A \mid\!\sim C$, which is interpreted by the probability assessment $P(C|A) = 1$. The negated conditional $\sim(A \to C)$ is then a negated default $A \mid\!\not\sim C$, which is interpreted by the wide scope negation of negating conditionals (see, e.g., Gilio & Sanfilippo, 2013c; Gilio et al., 2016):

Conditional	Default	Probabilistic interpretation
$A \to C$	$A \mid\!\sim C$	$P(C\|A) = 1$
$\sim(A \to C)$	$A \mid\!\not\sim C$	$P(C\|A) \neq 1$

We interpret the inner negation $\sim A$ by $\bar{A}$. Then, the conditional $A \to \sim C$ is interpreted by the default $A \mid\!\sim \bar{C}$. Likewise, $\sim A \to C$ is interpreted by $\bar{A} \mid\!\sim C$.

Approach 1: Validity as satisfaction of prob. constraints

We first consider non-iterated connexive principles, e.g., (AT), (AT'), (AB), (AS).

Definition:

A (non-iterated) connexive principle is valid if and only if the probabilistic constraint associated with the connexive principle is satisfied by every coherent assessment on the involved conditional events.

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Validity of (AT).
The negated conditional $\sim(\sim A \to A)$ is interpreted by the negated default $\bar{A} \not\mid\!\sim A$, that is $P(A|\bar{A}) \neq 1$. We observe that $P(A|\bar{A}) = 0$ is the unique precise coherent assessment on $A|\bar{A}$. Then, as $P(A|\bar{A}) = 0$ satisfies the probabilistic constraint $P(A|\bar{A}) \neq 1$, it follows that the connexive principle $\sim(\sim A \to A)$ is valid.

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Checking the validity of (AB) and (AS)

Validity of (AB).

We interpret $\sim((A \to B) \wedge (A \to \sim B))$ by the negation of $(A \mid\!\sim B, A \mid\!\sim \bar{B})$, that is $(P(B|A), P(\bar{B}|A)) \neq (1,1)$. This principle is valid because coherence requires that $P(B|A) + P(\bar{B}|A) = 1$.

Non-validity of (AS).

$\sim((A \to B) \wedge (\sim A \to B))$ is interpreted by the negation of $(A \mid\!\sim B, \bar{A} \mid\!\sim B)$, that is $(P(B|A), P(B|\bar{A})) \neq (1,1)$. (AS) is not valid because there exists a coherent probability assessment, see e.g., $(P(B|A) = 1, P(B|\bar{A}) = 1)$, which does not satisfy $(P(B|A), P(B|\bar{A})) \neq (1,1)$.

Approach 1: Checking iterated connexive principles

We interpret the main connective ($\rightarrow$) of iterated principles (e.g. (BT), (BT')) as a probabilistic consequence relation ($\Rightarrow$).

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An iterated connexive principle $\bigcirc \Rightarrow \square$ is valid if and only if the probabilistic constraint in the conclusion $\square$ is satisfied by every coherent extension from the premise $\bigcirc$ to the conclusion $\square$.

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Validity of (BT)

$(A \rightarrow B) \rightarrow {\sim}(A \rightarrow {\sim}B)$ is interpreted by $(P(B|A) = 1) \Rightarrow (P(\bar{B}|A) \neq 1)$, which is valid because if $P(B|A) = 1$, then $P(\bar{B}|A) = 1 - P(B|A) = 0 \neq 1$.

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Non-validity of (RBT) and (RBT')

(RBT): ${\sim}(A \rightarrow {\sim}B) \rightarrow (A \rightarrow B)$ is interpreted by $(P(\bar{B}|A) \neq 1) \;\Rightarrow\; (P(B|A) = 1)$, which is not valid because if $P(\bar{B}|A) \neq 1$, then $P(B|A) = 1 - P(\bar{B}|A) > 0$, but not necessary equal to 1 (e.g. $P(\bar{B}|A) = P(B|A) = 0.5$). Likewise, (RBT') ${\sim}(A \rightarrow B) \rightarrow (A \rightarrow {\sim}B)$ is not valid.

Approach 1: Checking validity of iterated connexive principles (cont.)

Non-validity of (B3)

We interpret $(A \to B) \to \sim(\sim A \to B)$ by $(P(B|A) = 1) \Rightarrow (P(B|\bar{A}) \neq 1)$, which is not valid because if $P(B|A) = 1$, then $P(B|\bar{A}) \in [0,1]$ and hence $P(B|\bar{A}) = 1$ is a coherent extension of $P(B|A) = 1$.

Non-validity of (B4)

We interpret $(\sim A \to B) \to \sim(A \to B)$ by $(P(B|\bar{A}) = 1) \Rightarrow (P(B|A) \neq 1)$, which is not valid because if $P(B|\bar{A}) = 1$, then $P(B|A) \in [0,1]$ and hence $P(B|A) = 1$ is a coherent extension of $P(B|\bar{A}) = 1$.

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Approach 2: Validity in terms of conditional random quantities

Instead of interpreting basic conditionals as defaults in terms of probabilistic constraints (Approach 1), we analyse connexive principles as conditional random quantities within the theory of logical operations among conditional events (Approach 2).

In Approach 2, a basic conditional is a conditional event (instead of a conditional probabilistic constraint) which is a three-valued object. Logical operations among conditional events yield (not conditional events, but) conditional random quantities with more than three possible values (Gilio & Sanfilippo, 2014).

Definition (Approach 2):
A connexive principle is valid if and only if the associated conditional random quantity is constant and equal to 1.

A conditional event as a random quantity

Given an event A we denote by the same symbol its indicator, which is equal to 1 or 0 according to whether A is true or false, respectively.

Let $p = P(A|H)$ be a conditional probability assessment on $A|H$. The indicator of $A|H$ (denoted by the same symbol) is defined as the random quantity (see, e.g., Gilio & Sanfilippo, 2014)

$$A|H = AH + p \cdot (1 - H) = AH + p \cdot \bar{H} = \begin{cases} 1, & \text{if } AH \text{ is true (win)}, \\ 0, & \text{if } \bar{A}H \text{ is true (lose)}, \\ p, & \text{if } \bar{H} \text{ is true (money back)}. \end{cases}$$

Of course, the third value of the random quantity $A|H$ (subjectively) depends on the assessed probability $P(A|H) = p$.

Approach 2: Validating (AT)

A conditional $A \to B$ is interpreted by $B|A$, where

$$B|A = AB + P(B|A)\bar{A}.$$

We interpret $\sim(A \to B)$ by $\overline{B|A}$.
The negation $\overline{B|A}$ of the conditional event $B|A$ is defined by

$$\overline{B|A} = 1 - B|A = (1 - B)|A = \bar{B}|A \tag{1}$$

because $P(\bar{B}|A) = 1 - P(B|A)$.
Aristotle's Thesis (AT)
We interpret $\sim(\sim A \to A)$ by $\overline{A|\bar{A}}$. Then, as $P(\bar{A}|\bar{A}) = 1$, it follows that

$$\overline{A|\bar{A}} = \bar{A}|\bar{A} = \bar{A} + P(\bar{A}|\bar{A})\bar{A} = A + \bar{A} = 1. \tag{2}$$

Therefore, (AT) is valid because $\overline{A|\bar{A}}$ is constant and equal to 1.

Validating (AT') and (AB)

Aristotle's Thesis' (AT')

We interpret $\sim(A \to \sim A)$ by $\overline{\bar{A}|A}$. Like in (AT), it holds that

$$\overline{\bar{A}|A} = \bar{\bar{A}}|A = A|A = 1, \tag{3}$$

which validates (AT').

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which validates (AT').

Abelard's Thesis (AB)

We interpret $\sim((A \to B) \wedge (A \to \sim B))$ by the conditional random quantity $\overline{(B|A) \wedge (\bar{B}|A)}$, where $A \neq \varnothing$. We observe that $(B|A) \wedge (\bar{B}|A) = (B \wedge \bar{B})|A = \varnothing|A$. Then,

$$\overline{(B|A) \wedge (\bar{B}|A)} = \overline{\varnothing|A} = \bar{\varnothing}|A = \Omega|A = A|A = 1, \tag{4}$$

which validates (AB).

Conjoined conditionals as a random quantity

Definition

Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their conjunction as

$$(A|H) \wedge (B|K) = (ABHK + x\bar{H}BK + y\bar{K}AH)|(H \vee K) =$$

$$= \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ & \\ & \\ & \end{cases}$$

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where $z = \mathbb{P}[(A|H) \wedge (B|K)]$. Of course $(A|H) \wedge (B|H) = AB|H$.

Conjoined conditionals as a random quantity

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- 1, if all conditional events are true;

Conjoined conditionals as a random quantity

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where $z = \mathbb{P}[(A|H) \wedge (B|K)]$. Of course $(A|H) \wedge (B|H) = AB|H$.
In the betting framework you agree to pay $z = \mathbb{P}[(A|H) \wedge (B|K)]$ with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;

Conjoined conditionals as a random quantity

Definition

Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their conjunction as

$$(A|H) \wedge (B|K) = (ABHK + x\bar{H}BK + y\bar{K}AH)|(H \vee K) =$$
$$= \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \bar{A}H \text{ is true or } \bar{B}K \text{ is true,} \\ x = P(A|H), & \text{if } \bar{H}BK \text{ is true,} \\ y = P(B|K), & \text{if } \bar{K}AH \text{ is true,} \\ z = \mathbb{P}[(A|H) \wedge (B|K)], & \text{if } \bar{H}\,\bar{K} \text{ is true,} \end{cases}$$

where $z = \mathbb{P}[(A|H) \wedge (B|K)]$. Of course $(A|H) \wedge (B|H) = AB|H$.
In the betting framework you agree to pay $z = \mathbb{P}[(A|H) \wedge (B|K)]$ with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of that conditional event which is void, if one conditional event is void and the other one is true;

Conjoined conditionals as a random quantity

Definition

Given any pair of conditional events $A|H$ and $B|K$, with $P(A|H) = x$, $P(B|K) = y$, we define their conjunction as

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where $z = \mathbb{P}[(A|H) \wedge (B|K)]$. Of course $(A|H) \wedge (B|H) = AB|H$.
In the betting framework you agree to pay $z = \mathbb{P}[(A|H) \wedge (B|K)]$ with the proviso that you will receive:

- 1, if all conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of that conditional event which is void, if one conditional event is void and the other one is true;
- the quantity z that you paid, if both conditional events are void.

Non-validity of Aristotle's Second Thesis (AS)

We interpret $\sim((A \to B) \wedge (\sim A \to B))$ by the random quantity $\overline{(B|A) \wedge (B|\bar{A})}$. By setting $P(B|A) = x$ and $P(B|\bar{A}) = y$, it follows that (Gilio & Sanfilippo, 2013b, 2019a)

$$(B|A) \wedge (B|\bar{A}) = (B|A) \cdot (B|\bar{A}) = \begin{cases} 0, & \text{if } \bar{B} \text{ is } \textit{true}, \\ y, & \text{if } AB \text{ is } \textit{true}, \\ x, & \text{if } \bar{A}B \text{ is } \textit{true}. \end{cases}$$

Then,

$$\overline{(B|A) \wedge (B|\bar{A})} = 1 - (B|A) \wedge (B|\bar{A}) = \begin{cases} 1, & \text{if } \bar{B} \text{ is } \textit{true}, \\ 1-y, & \text{if } AB \text{ is } \textit{true}, \\ 1-x, & \text{if } \bar{A}B \text{ is } \textit{true}, \end{cases}$$

which is not constant and can therefore not necessarily be equal to 1.

In particular, by choosing the coherent assessment $x = y = 1$, it follows that

$$\overline{(B|A) \wedge (B|\bar{A})} = \bar{B},$$

which is not necessary equal to 1 as it could be either 1 or 0, according to whether $\bar{B}$ is true or false, respectively.
Therefore, (AS) is not valid.

Iterated conditioning

We recall that

$$A|H = A \wedge H + p\bar{H},$$

where $p = P(A|H)$.
Given any pair of conditional events $A|H$ and $B|K$, the notion of an iterated conditional $(B|K)|(A|H)$ is based on the same structure, i.e.

$$\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc},$$

where $\square$ denotes $B|K$ and $\bigcirc$ denotes $A|H$.

Iterated conditioning

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where $\square$ denotes $B|K$ and $\bigcirc$ denotes $A|H$.

In betting terms, $\mu = \mathbb{P}(\square|\bigcirc)$ is the amount that you agree to pay, with the proviso that you receive the random quantity $\square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}$.

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We recall that

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Given any pair of conditional events $A|H$ and $B|K$, the notion of an iterated conditional $(B|K)|(A|H)$ is based on the same structure, i.e.

$$\square|\bigcirc = \square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc},$$

where $\square$ denotes $B|K$ and $\bigcirc$ denotes $A|H$.

In betting terms, $\mu = \mathbb{P}(\square|\bigcirc)$ is the amount that you agree to pay, with the proviso that you receive the random quantity $\square \wedge \bigcirc + \mathbb{P}(\square|\bigcirc)\bar{\bigcirc}$. Then, the notion of the iterated conditional $(B|K)|(A|H)$ is defined by (see, e.g., Gilio & Sanfilippo, 2013a, 2013b, 2014):

Definition

Given any pair of conditional events $A|H$ and $B|K$, with $AH \neq \emptyset$, we define the iterated conditional $(B|K)|(A|H)$ as

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu\bar{A}|H\,,$$

where μ is the prevision of $(B|K)|(A|H)$.

In explicit terms

Operatively, if $P(A|H) = x, P(B|K) = y, \mathbb{P}[(A|H) \wedge (B|K)] = z$, then $\mu = \mathbb{P}[(B|K)|(A|H)]$ represents the amount you agree to pay, with the proviso that you will receive the quantity

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu \bar{A}|H = \begin{cases} 1, & AHBK, \\ 0, & AH\bar{B}K, \\ y, & AH\bar{K}\ , \\ \mu & \bar{A}H, \\ x + \mu(1-x), & \bar{H}BK, \\ \mu(1-x), & \bar{H}\bar{B}K, \\ z + \mu(1-x), & \bar{H}\bar{K}\,. \end{cases}$$

We observe that by the linearity of prevision

$$\mu = \mathbb{P}[(B|K)|(A|H)] = \mathbb{P}[(B|K) \wedge (A|H)] + \mu P(\bar{A}|H) = z + \mu(1-x).$$

Validity of Boethius' Thesis (BT)

We interpret $(A \to B) \to \sim(A \to \sim B)$ by the iterated conditional $(\overline{\bar{B}|A})|(B|A)$, with $AB \neq \emptyset$. We recall that $(\overline{\bar{B}|A}) = B|A$ and that $(B|A) \wedge (B|A) = B|A$. Then, $(\overline{\bar{B}|A})|(B|A) = (B|A)|(B|A)$. Moreover, by setting $P(B|A) = x$ and $\mathbb{P}[(B|A)|(B|A)] = \mu$, it holds that

$$(B|A)|(B|A) = (B|A) + \mu(1 - B|A) = \begin{cases} 1, & \text{if } AB \text{ is } \textit{true}, \\ \mu, & \text{if } A\bar{B}, \text{ is } \textit{true}, \\ x + \mu(1-x), & \text{if } \bar{A} \text{ is } \textit{true}. \end{cases}$$

By the linearity of prevision it holds that

$$\mu = \mathbb{P}[(B|A)|(B|A)] = P(B|A) + \mu P(\bar{B}|A) = x + \mu(1-x).$$

Then,

$$(B|A)|(B|A) = \begin{cases} 1, & \text{if } AB \text{ is } \textit{true}, \\ \mu, & \text{if } \bar{A} \vee \bar{B} \text{ is } \textit{true}. \end{cases}$$

Then, by coherence it must be $\mu = 1$ and hence (Gilio & Sanfilippo, 2014, Remark 2)

$$(B|A)|(B|A) = 1. \tag{5}$$

Therefore $(\overline{\bar{B}|A})|(B|A)$ is constant and equal to 1, which validates (BT).

Validity of (BT'), (RBT), and (RBT')

Boethius' Thesis' (BT')

We intrepret $(A \to {\sim}B) \to {\sim}(A \to B)$ by the iterated conditional $(\overline{B|A})|(\bar{B}|A)$, where $A\bar{B} \neq \emptyset$. By observing that $\overline{B|A} = \bar{B}|A$, it follows that $(\overline{B|A})|(\bar{B}|A) = (\bar{B}|A)|(\bar{B}|A)$ which is constant and equal to 1 because of (5). Therefore (BT') is validated.

Reversed Boethius' Thesis (RBT)

We interpret ${\sim}(A \to {\sim}B) \to (A \to B)$ by the iterated conditional $(B|A)|(\overline{\bar{B}|A})$, where $AB \neq \emptyset$. As $(\overline{\bar{B}|A}) = B|A$, it follows from (5) that $(B|A)|(\overline{\bar{B}|A}) = (B|A)|(B|A) = 1$. Therefore (RBT) is validated.

Reversed Boethius' Thesis' (RBT')

We interpret ${\sim}(A \to B) \to (A \to {\sim}B)$ by the iterated conditional $(\bar{B}|A)|(\overline{B|A})$, where $A\bar{B} \neq \emptyset$. As $(\overline{B|A}) = \bar{B}|A$, it follows from (5) that $(\bar{B}|A)|(\overline{B|A}) = (\bar{B}|A)|(\bar{B}|A) = 1$. Therefore (RBT') is validated.

Non-Validity of (B3) and (B4)

Boethius Variation (B3)

We interpret $(A \to B) \to \sim(\sim A \to B)$ by the iterated conditional $(\overline{B|\bar{A}})|(B|A)$. We observe that $(\overline{B|\bar{A}})|(B|A) = (\bar{B}|\bar{A})|(B|A)$, because $\overline{B|\bar{A}} = \bar{B}|\bar{A}$. By setting $P(B|A) = x$, $P(\bar{B}|\bar{A}) = y$, and $\mathbb{P}[(\bar{B}|\bar{A})|(B|A)] = \mu$, it holds that

$$(\bar{B}|\bar{A})|(B|A) = (\bar{B}|\bar{A}) \wedge (B|A) + \mu(1 - B|A) = \begin{cases} y, & \text{if } AB \text{ is } \textit{true}, \\ \mu, & \text{if } A\bar{B} \text{ is } \textit{true}, \\ \mu(1-x), & \text{if } \bar{A}B \text{ is } \textit{true}, \\ x + \mu(1-x), & \text{if } \bar{A}\bar{B} \text{ is } \textit{true}, \end{cases}$$

which is not constant and can therefore not necessarily be equal to 1. For example, if we choose the coherent assessment $x = y = 1$, it follows that

$$(\bar{B}|\bar{A})|(B|A) = (\bar{B}|\bar{A}) \wedge (B|A) + \mu(1 - B|A) = \begin{cases} 1, & \text{if } AB \text{ is } \textit{true}, \\ \mu, & \text{if } A\bar{B}, \text{ is } \textit{true}, \\ 0, & \text{if } \bar{A}B \text{ is } \textit{true}, \\ 1, & \text{if } \bar{A}\bar{B} \text{ is } \textit{true}, \end{cases}$$

which is not constant and equal to 1. Therefore (B3) is invalid.

Boethius Variation (B4)

We interpret $(\sim A \to B) \to \sim(A \to B)$ by the iterated conditional $(\overline{B|A})|(B|\bar{A})$. We observe that $(\overline{B|A})|(B|\bar{A})$ is not constant and not necessary equal to 1 because it is equivalent to (B3) when A is replaced by $\bar{A}$. Therefore, (B4) is not valid.

Connex. princ. interpreted by compounds of cond. events

Name	Connexive principle	Interpretation	Validity
(AT)	$\sim(\sim A \to A)$	$\overline{A\vert\bar{A}} = 1$	yes
(AT')	$\sim(A \to \sim A)$	$\overline{\bar{A}\vert A} = 1$	yes
(AB)	$\sim((A \to B) \wedge (A \to \sim B))$	$\overline{(B\vert A) \wedge (\bar{B}\vert A)} = 1$	yes
(AS)	$\sim((A \to B) \wedge (\sim A \to B))$	$\overline{(B\vert A) \wedge (B\vert\bar{A})} \neq 1$	no
(BT)	$(A \to B) \to \sim(A \to \sim B)$	$\overline{(\bar{B}\vert A)}\vert(B\vert A) = 1$	yes
(BT')	$(A \to \sim B) \to \sim(A \to B)$	$\overline{(B\vert A)}\vert(\bar{B}\vert A) = 1$	yes
(RBT)	$\sim(A \to \sim B) \to (A \to B)$	$(B\vert A)\vert\overline{(\bar{B}\vert A)} = 1$	yes
(RBT')	$\sim(A \to B) \to (A \to \sim B)$	$(\bar{B}\vert A)\vert\overline{(B\vert A)} = 1$	yes
(B3)	$(A \to B) \to \sim(\sim A \to B)$	$\overline{(B\vert\bar{A})}\vert(B\vert A) \neq 1$	no
(B4)	$(\sim A \to B) \to \sim(A \to B)$	$\overline{(B\vert A)}\vert(B\vert\bar{A}) \neq 1$	no

Connex. princ. interpreted by compounds of cond. events

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(AB)	$\sim((A \to B) \wedge (A \to \sim B))$	$\overline{(B\vert A) \wedge (\bar{B}\vert A)} = 1$	yes
(AS)	$\sim((A \to B) \wedge (\sim A \to B))$	$\overline{(B\vert A) \wedge (B\vert\bar{A})} \neq 1$	no
(BT)	$(A \to B) \to \sim(A \to \sim B)$	$\overline{(\bar{B}\vert A)}\vert(B\vert A) = 1$	yes
(BT')	$(A \to \sim B) \to \sim(A \to B)$	$\overline{(B\vert A)}\vert(\bar{B}\vert A) = 1$	yes
(RBT)	$\sim(A \to \sim B) \to (A \to B)$	$(B\vert A)\vert\overline{(\bar{B}\vert A)} = 1$	yes
(RBT')	$\sim(A \to B) \to (A \to \sim B)$	$(\bar{B}\vert A)\vert\overline{(B\vert A)} = 1$	yes
(B3)	$(A \to B) \to \sim(\sim A \to B)$	$\overline{(B\vert\bar{A})}\vert(B\vert A) \neq 1$	no
(B4)	$(\sim A \to B) \to \sim(A \to B)$	$\overline{(B\vert A)}\vert(B\vert\bar{A}) \neq 1$	no

Note that (RBT) and (RBT') are not valid in Approach 1.

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Name	Connexive principle	Interpretation	Validity
(AT)	$\sim(\sim A \to A)$	$\overline{A\vert\bar{A}} = 1$	yes
(AT')	$\sim(A \to \sim A)$	$\overline{\bar{A}\vert A} = 1$	yes
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(AS)	$\sim((A \to B) \wedge (\sim A \to B))$	$\overline{(B\vert A) \wedge (B\vert\bar{A})} \neq 1$	no
(BT)	$(A \to B) \to \sim(A \to \sim B)$	$\overline{(\bar{B}\vert A)}\vert(B\vert A) = 1$	yes
(BT')	$(A \to \sim B) \to \sim(A \to B)$	$\overline{(B\vert A)}\vert(\bar{B}\vert A) = 1$	yes
(RBT)	$\sim(A \to \sim B) \to (A \to B)$	$(B\vert A)\vert\overline{(\bar{B}\vert A)} = 1$	yes
(RBT')	$\sim(A \to B) \to (A \to \sim B)$	$(\bar{B}\vert A)\vert\overline{(B\vert A)} = 1$	yes
(B3)	$(A \to B) \to \sim(\sim A \to B)$	$\overline{(B\vert\bar{A})}\vert(B\vert A) \neq 1$	no
(B4)	$(\sim A \to B) \to \sim(A \to B)$	$\overline{(B\vert A)}\vert(B\vert\bar{A}) \neq 1$	no

Note that (RBT) and (RBT') are not valid in Approach 1. If, however, we use the narrow scope negation in Approach 1, we can validate (RBT) and (RBT') in Approach 1.

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- Connexivity is not only psychologically plausible (e.g., Pfeifer, 2012b; Pfeifer & Tulkki, 2017) but also emerges naturally under probabilistic interpretations
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- Aristotle's Second Thesis, (B3), and (B4) are not valid in both approaches
- Coherence-based probability logic allows for bridging qualitative and quantitative reasoning, applications include, e.g.,
  - Aristotelian syllogisms (e.g., Pfeifer & Sanfilippo, 2018, 2019, m.s.)
  - Squares and hexagons of oppositions in terms of probability and defaults (e.g., Pfeifer & Sanfilippo, 2017)
  - Nonmonotonic reasoning (System P (Gilio, 2002; Gilio & Sanfilippo, 2019b), Weak Transitivity (Gilio et al., 2016))
  - Towards a deduction theorem (Gilio et al., 2020)
  - Paradoxes of the material conditional (Pfeifer, 2014, in press)
  - Psychology of reasoning (e.g., Pfeifer, 2013; Pfeifer & Kleiter, 2009, 2010)
  - Iterated conditionals and compounds ...
- Future work: analysis of other connexive principles (suggestions welcome!)

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